

BASIC MATHEMATICS

COURSE SYLLABUS

I. NATURE AND PURPOSE OF THE COURSE

Basic Mathematics is an introductory course in the foundations, language, methods, and principal elementary structures of mathematics. It is not conceived merely as a review of arithmetic, nor as a collection of computational devices preliminary to algebra. Its purpose is to establish those habits of exact thought upon which the subsequent study of pure and applied mathematics necessarily depends, and to carry the student, in an orderly progression, from elementary mathematical reasoning and the real number system through the fundamental ideas of algebra, geometry, probability, linear methods, functions, limits, differentiation, antiderivatives, and the definite integral.

Particular attention is given from the beginning to axiomatic reasoning, logical symbols and operators, definition, proof, mathematical notation, and the distinction between calculation and demonstration. These matters are not isolated preliminary topics: they constitute the language and method by which the whole course is studied.

The student will be expected not merely to obtain correct answers, but to understand the mathematical statements being made, the assumptions upon which an argument depends, and the reasons by which one conclusion follows from another.

THE GUIDING PRINCIPLE OF THE COURSE MAY BE STATED SIMPLY:

MATHEMATICS IS AN ORDERED DEDUCTIVE SCIENCE.

II. GENERAL COURSE OF STUDY

The following subjects constitute the ordinary field of the course. Definitions, principles, propositions, computations, constructions, proofs, and methods arising from any of these subjects may be examined.

A. FOUNDATIONS, AXIOMATIC REASONING, AND MATHEMATICAL LOGIC

1. The nature of mathematical abstraction; concrete and mathematical objects
2. Statements and propositions; truth and falsity
3. Logical symbols and operators: negation, conjunction, disjunction, implication, and equivalence
4. Truth tables, tautologies, contradictions, and logical equivalence
5. Converse, inverse, and contrapositive
6. Universal and existential quantifiers; negation of quantified statements
7. Assumptions, conclusions, necessary conditions, and sufficient conditions
8. Definitions, undefined terms, axioms, postulates, propositions, and theorems
9. Direct reasoning, indirect reasoning, and proof by contradiction
10. Counterexamples and the distinction between evidence and proof
11. Elementary rules of inference and the structure of valid arguments
12. The elementary structure of an axiomatic system; consistency and logical consequence

B. NUMBER SYSTEMS AND THE LAWS OF NUMBER

1. The set of integers and the set of rational numbers
2. Fractions, ratios, decimal representations, and rational arithmetic
3. The set of real numbers; irrational numbers and the real number line
4. Order, intervals, absolute value, and inequalities
5. Commensurable and incommensurable magnitudes; the significance of $\sqrt{2}$

6. Closure, identity elements, inverses, and the commutative, associative, and distributive laws
7. Powers, roots, exponents, scientific notation, and numerical estimation
8. Divisibility, primes, factorization, greatest common divisor, and least common multiple

C. ELEMENTS OF SET THEORY

1. Sets, elements, membership, and set-builder notation
2. Equality of sets, subsets, proper subsets, and the empty set
3. Finite and infinite sets
4. Union, intersection, difference, complements, and disjoint sets
5. Venn diagrams and elementary set identities
6. Ordered pairs and Cartesian products
7. Sets as a language for mathematical structure

D. COMBINATORICS

1. The fundamental counting principle
2. Factorials
3. Permutations
4. Combinations
5. Counting with and without repetition
6. Elementary applications of combinatorial reasoning

E. PROBABILITY

1. Experiments, outcomes, sample spaces, and events
2. Probability of an event
3. Complementary events
4. Union and intersection of events
5. Conditional probability in elementary cases
6. Independent events
7. Applications of counting methods to probability

F. DESCRIPTIVE STATISTICS

1. Data, variables, and frequency distributions
2. Tables and graphical representations
3. Arithmetic mean, median, and mode
4. Range and elementary measures of dispersion
5. Variance and standard deviation in elementary applications
6. Interpretation of numerical and graphical information
7. The distinction between mathematical description and statistical inference

G. SYSTEMS OF LINEAR EQUATIONS

1. Linear equations in two and three variables
2. Systems of linear equations
3. Consistent and inconsistent systems
4. Unique solutions and infinitely many solutions
5. Algebraic and graphical solution of elementary systems
6. Equivalent systems and elimination methods

H. MATRICES AND DETERMINANTS

1. Definition, notation, order, and equality of matrices
2. Row, column, square, zero, diagonal, and identity matrices
3. Matrix addition, subtraction, and scalar multiplication
4. Matrix multiplication and its non-commutativity
5. Transpose of a matrix
6. Determinants of 2×2 and 3×3 matrices
7. Minors, cofactors, and elementary determinant expansion
8. Singular and nonsingular matrices; invertibility

9. Inverse of a 2×2 matrix and elementary matrix equations

I. SOLVING SYSTEMS OF LINEAR EQUATIONS

1. Augmented matrices
2. Elementary row operations
3. Row-echelon and reduced row-echelon forms
4. Gaussian elimination and Gauss-Jordan elimination
5. Matrix methods for systems of linear equations
6. Cramer's rule in elementary cases
7. Interpretation of determinant conditions for solvability

J. ELEMENTS OF GEOMETRY

1. Point, line, plane, segment, ray, and angle
2. Parallelism, perpendicularity, congruence, and similarity
3. Triangles, quadrilaterals, circles, perimeter, and area
4. The Pythagorean theorem
5. Elementary geometric constructions
6. Euclidean axiomatic reasoning
7. The distinction between a diagram and a proof
8. Classical construction problems and the role of mathematical impossibility

K. TRIGONOMETRY

1. Measurement of angles; degrees and radians
2. Right-triangle trigonometry
3. Sine, cosine, and tangent
4. Reciprocal trigonometric ratios
5. Elementary trigonometric identities
6. Solution of right and oblique triangles in elementary cases
7. Laws of sines and cosines

L. ANALYTIC GEOMETRY

1. The Cartesian coordinate system
2. Distance and midpoint formulas
3. Slope and equations of straight lines
4. Parallel and perpendicular lines
5. Graphs of elementary equations
6. Intersection of curves in elementary cases

M. VECTORS

1. Vectors and scalars
2. Magnitude and direction
3. Vector addition and subtraction
4. Scalar multiplication
5. Components of vectors
6. Unit vectors
7. Elementary applications of vectors in the plane

N. COMPLEX NUMBERS

1. The imaginary unit and complex numbers
2. Real and imaginary parts
3. Equality, addition, subtraction, multiplication, and division
4. Complex conjugates
5. Modulus and elementary geometric representation
6. Polar form in elementary cases
7. Powers and roots in elementary cases

O. VECTORS IN SPACE

1. Three-dimensional Cartesian coordinates
2. Vectors in $\mathbb{R}^3$
3. Magnitude, direction, and components
4. Dot product and orthogonality
5. Elementary cross product
6. Lines in space
7. Elementary geometric applications

P. PLANE ANALYTIC GEOMETRY

1. Coordinate description of geometric loci
2. Straight lines and families of lines
3. The circle
4. Parabolas
5. Ellipses
6. Hyperbolas
7. Translation and elementary analysis of conic equations

Q. ANALYTIC GEOMETRY IN SPACE

1. Coordinates and distance in three dimensions
2. Equations of planes
3. Lines and planes in space
4. Angles and elementary distance relations
5. Spheres
6. Elementary surfaces and their equations

R. SEQUENCES

1. Definition and notation of a sequence
2. Explicit and recursive descriptions
3. Arithmetic sequences
4. Geometric sequences
5. Monotone and bounded sequences
6. Elementary algebra of sequences

S. LIMITS OF SEQUENCES

1. Intuitive and mathematical meaning of convergence
2. Limits of elementary sequences
3. Limit laws for sequences
4. Divergent sequences
5. Infinite behavior in elementary cases
6. The role of limits in the transition to analysis

T. FUNCTIONS

1. Relations and functions
2. Domain, codomain, and range
3. Function notation and evaluation
4. Graphs, tables, formulas, and verbal descriptions
5. Composition of functions
6. One-to-one functions and inverse functions
7. Transformations and elementary families of functions

U. POLYNOMIALS

1. Polynomial expressions and polynomial functions
2. Degree, coefficients, and zeros
3. Addition, subtraction, multiplication, and division of polynomials
4. Factoring

5. Remainder and factor principles in elementary form
6. Graphs and qualitative behavior of polynomial functions
7. Rational functions as quotients of polynomials

V. LIMITS OF FUNCTIONS

1. The concept of a limit
2. One-sided limits
3. Two-sided limits
4. Limit laws
5. Infinite limits and limits at infinity in elementary cases
6. Algebraic evaluation of limits
7. Indeterminate forms in elementary examples

W. CONTINUOUS FUNCTIONS

1. Continuity at a point
2. Continuity on an interval
3. Algebra of continuous functions
4. Discontinuities
5. Continuity of polynomial, rational, trigonometric, exponential, and logarithmic functions where defined
6. Elementary consequences of continuity

X. DERIVATIVES OF FUNCTIONS

1. Derivative as a limit of a difference quotient
2. Geometric meaning: slope of the tangent line
3. Rate-of-change interpretation
4. Derivative notation
5. Derivatives of constants, powers, sums, products, and quotients
6. Chain rule
7. Derivatives of polynomial and rational functions
8. Derivatives of elementary trigonometric functions
9. Derivatives of exponential and logarithmic functions
10. Higher derivatives in elementary cases

Y. REPRESENTING FUNCTIONS AND MODELS

1. Increasing and decreasing functions
2. Critical points
3. Local maxima and minima
4. Concavity and points of inflection
5. Elementary curve sketching using derivatives
6. Optimization in elementary models
7. Rates of change and mathematical modeling
8. Interpretation of parameters and graphical behavior

Z. TRIGONOMETRIC FUNCTIONS

1. Trigonometric functions on the real line
2. The unit circle
3. Graphs, periods, amplitudes, and phase shifts
4. Fundamental trigonometric identities
5. Inverse trigonometric functions in elementary cases
6. Trigonometric equations
7. Limits and derivatives involving trigonometric functions

AA. LOGARITHMIC AND EXPONENTIAL FUNCTIONS

1. Exponential functions and their properties
2. The natural exponential function

3. Logarithms and logarithmic laws
4. The natural logarithm
5. Inverse relation between exponential and logarithmic functions
6. Exponential growth and decay
7. Limits and derivatives of exponential and logarithmic functions
8. Elementary applications and models

AB. ANTIDERIVATIVES OF A FUNCTION

1. Antiderivatives and indefinite integration
2. Families of antiderivatives and the constant of integration
3. Elementary antiderivative formulas
4. Antiderivatives of powers
5. Elementary antiderivatives involving exponential and trigonometric functions
6. Verification of antiderivatives by differentiation

AC. ANTIDERIVATIVES IN GENERAL

1. Linearity of antidifferentiation
2. Substitution in elementary cases
3. Antiderivatives arising from algebraic simplification
4. Initial-value problems in elementary form
5. Antiderivatives as inverse problems to differentiation
6. Elementary applications to motion and accumulation

AD. THE DEFINITE INTEGRAL

1. Area and accumulation as motivation
2. Partitions and Riemann sums at an introductory level
3. Definition and notation of the definite integral
4. Basic properties of the definite integral
5. Geometric interpretation and signed area
6. The Fundamental Theorem of Calculus
7. Evaluation of definite integrals by antiderivatives
8. Elementary applications of the definite integral

III. THE THIRTY PRINCIPAL TOPICS

For purposes of course organization, the principal mathematical subject matter may be summarized under the following thirty headings. The foundational work in logic, notation, axiomatic reasoning, proof, and mathematical writing accompanies these topics throughout.

- 1) The set of integers and rational numbers
- 2) The set of real numbers
- 3) Elements of set theory
- 4) Combinatorics
- 5) Probability
- 6) Descriptive Statistics
- 7) Systems of Linear Equations
- 8) Matrices and Determinants
- 9) Solving systems of linear equations
- 10) Elements of geometry
- 11) Trigonometry
- 12) Analytic Geometry
- 13) Vectors
- 14) Complex Numbers
- 15) Vectors in space
- 16) Plane Analytic Geometry
- 17) Analytic Geometry in Space

- 18) Sequences
- 19) Limits of Sequences
- 20) Functions
- 21) Polynomials
- 22) Limits of Functions
- 23) Continuous Functions
- 24) Derivatives of Functions
- 25) Representing functions and models
- 26) Trigonometric Functions
- 27) Logarithmic and Exponential Functions
- 28) Antiderivatives of a function
- 29) Antiderivatives in general
- 30) The definite integral

IV. HISTORICAL AND FOUNDATIONAL THEMES

Historical questions will accompany the mathematical work whenever they illuminate the origin or meaning of an idea. Among these are the development of mathematics from counting and measurement; the rise of abstraction and deductive reasoning in Greek mathematics; the Pythagorean discovery of incommensurable magnitudes; Zeno's paradoxes and the problems of infinity and continuity; the axiomatic organization of geometry by Euclid; Aristotle's analysis of logical inference; the search for a precise language of reasoning; and the algebraic treatment of logic associated with George Boole.

These matters are not introduced as historical decoration. They serve to show why mathematics came to require definition, abstraction, rigor, and deduction, and why the ideas of limit, continuity, derivative, and integral belong to a long development of mathematical precision rather than to a mere collection of computational techniques.

V. STANDARD OF WORK

Students are expected to write mathematics in a clear and orderly manner. An answer unsupported by intelligible mathematical work may receive little or no credit even when the numerical result is correct. Symbols must be used according to their accepted meanings; variables must be identified; equalities must be genuine equalities; conclusions must follow from stated premises; and any theorem, definition, or property invoked in an argument must be used correctly.

Memorized rules are not substitutes for mathematical justification.

Statements such as "move it to the other side," "cross multiply," or "flip the fraction" will not, by themselves, constitute explanations.

THE STUDENT SHOULD BE PREPARED TO ANSWER NOT ONLY "WHAT IS THE ANSWER?" BUT ALSO "WHY IS IT TRUE?"

VI. EXAMINATIONS AND GRADING

The final grade in the course shall be determined entirely by formal examination.

MIDTERM EXAMINATION	50%
FINAL EXAMINATION	50%
TOTAL	100%

The midterm examination will ordinarily concern the material completed during the first portion of the course. The final examination may draw upon the whole mathematical language

and method developed during the course and will place particular emphasis upon the material studied after the midterm. Earlier ideas may therefore reappear whenever their use is necessary for later mathematics.

Exercises may be assigned throughout the course and are expected to be completed conscientiously. Their purpose is preparation and mastery. They do not replace examination.

VII. GENERAL EXAMINATION PRINCIPLE

Unless the instructor expressly excludes a topic, any definition, principle, computation, proof, notation, construction, theorem, or method appearing in the assigned course of study is FAIR GAME FOR EXAMINATION.

STUDENTS SHOULD THEREFORE AVOID THE MODERN AND UNFORTUNATE HABIT OF ASKING, "WILL THIS BE ON THE TEST?"

THE PROPER QUESTION IS, "DO I UNDERSTAND THE MATHEMATICS?"

VIII. FINAL REMARK

The title BASIC MATHEMATICS must not be mistaken for a promise that the course will demand little thought. The elementary foundations of a subject are often precisely those matters which require the greatest care. In this course, 'basic' refers not to triviality but to what is regarded as basic equipment for serious subsequent mathematical study.

The purpose of this course is not to hurry the student toward advanced mathematics. It is to make the student fit to study advanced mathematics. The course therefore begins with number, logic, set, and proof, and proceeds through the elementary structures required to reach the central ideas of the differential and integral calculus.

Before one proceeds farther, one should know how mathematical statements are formed, how definitions govern mathematical objects, how conclusions follow from assumptions, how number systems and functions are organized, how symbols are used, and why a proof is something altogether different from a collection of examples.

A SOUND FOUNDATION PERMITS ADVANCEMENT.

AN UNSOUND FOUNDATION MERELY PERMITS HEIGHT.

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CITY TUTORING MATHEMATICS ACADEMY

DEFINITION -- LOGIC -- PROOF -- REASON